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L^0-Projective Topologies on Linear Spaces

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For any finite measure space (S, M, v) let $L^0(S, M, v)$ denote the family of v-equivalence classes of measurable functions on S, endowed with the topology of convergence in measure. For any topological vector space $(X, \mathscr{G})$ let $\lambda^0(X, \mathscr{G})$ denote the family of all continuous linear maps on X into (possibly different) spaces $L^0(S, M, v)$. If $\mathscr{G}$ coincides with the coarsest topology on X making every map from $\lambda^0(X, \mathscr{G})$ continuous we call $(X, \mathscr{G})$ an L^0-projective space. It is shown that L^p-spaces, $0 < p \leqslant 2$, and hence any projective limit of such spaces, are all L^0-projective. It is further shown that l^p-spaces, $2 < p < \infty$, and the space c_0, are not L^0-projective.

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1. L^0-PROJECTIVE SPACES

For any vector space X, and family $\mathscr{L}$ of linear maps on X to possibly different topological vector spaces, $\mathscr{T}(\mathscr{L})$ is the coarsest topology on X making each map from $\mathscr{L}$ continuous (the projective topology). For any topological vector space $(X, \mathscr{G})$, X' is the topological dual.

DEFINITION. A topological vector space $(X, \mathscr{G})$ is L^0-projective iff $\mathscr{T}(\lambda^0(X, \mathscr{G})) = \mathscr{G}$. $(X, \mathscr{G})$ is an L^0-subspace iff there exists a linear homeomorphism from X onto a subspace of some $L^0(S, M, v)$.

In the terminology of L. Schwartz (10, 11), a space is L^0-projective iff it has an 0-cotypical continuous map into some L^0. $_a$

We make the following observations: Every $L^0(S, M, v)$ is a topological vector space; hence, so also is $(X, \mathscr{T}(\lambda^0(X, \mathscr{G}))$ for any topological vector space $(X, \mathscr{G})$ (H. Schaefer, 9, p. 51). Clearly,

$$\mathscr{T}(\lambda^0(X, \mathscr{G})) \subset \mathscr{G}.$$

Also, by assigning to each $f \in X'$ the corresponding unit mass on X', one checks that $\mathscr{T}(\lambda^0(X, \mathscr{G}))$ is finer than the weak topology on X.

In a straightforward manner one can prove also the following assertions.

THEOREM 1. *Every L^0-subspace is L^0-projective. A normed space is an L^0-subspace iff it is an L^0-projective space.*

THEOREM 2. *For any family $\mathscr{L}$ of linear maps from a vector space X to L^0-projective spaces, $\mathscr{T}(\mathscr{L})$ is L^0-projective. In particular, any subspace, Cartesian product, or projective limit of L^0-projective spaces is L^0-projective.*

2. Linear Stochastic Processes

Without loss of generality we shall assume our measure spaces to be probability measure spaces. A linear map from a vector space X into an $L^0(S, M, v)$ is then called a *linear stochastic process*. We recall briefly certain well-known facts about such processes: cf. e.g. Gelfand and Vilenkin [5], Badrikian [1], Fernique [4], Cartier [3], Waldenfels [13].

For any linear process $A: X \to L^0(S, M, v)$, the *characteristic function* φ of A is defined by

$$\varphi(x) = \int e^{iA(x)(s)}\, dv(s), \qquad x \in X.$$

Bochner's theorem has the following well-known generalization: A complex-valued function φ on a real vector space is the characteristic function of some linear process iff φ is positive-definite, $\varphi(0) = 1$, and φ is continuous on finite dimensional subspaces. It is also known that if φ is the characteristic function of a linear process A on a topological vector space $(X, \mathscr{G})$ then φ is continuous iff A is continuous. Hence, THEOREM 3. *A topological vector space $(X, \mathscr{G})$ is L^0-projective if there exists a positive-definite function φ on X such that*

$$x \to 0 \text{ in } X \qquad \text{iff} \quad \varphi(x) \to 0.$$

For any measure space (S, M, v), finite or not, it is known that for $0 < p \leqslant 2$, $\varphi: f \in L^p(v) \to \exp(-\int |f|^p \, dv) \in \mathbb{C}$ is a positive definite-function. [2, Thm. 1, *p.* 232].

Thus, by Thms. 2, 3,

THEOREM 4. *Any L^p-space for $0 < p \leqslant 2$ is an L^0-subspace. For any vector space X and family $\mathscr{L}$ of linear maps from X into L^p-spaces, $0 < p \leqslant 2$, $\mathscr{T}(\mathscr{L})$ is an L^0-projective topology on X.*

A space with a topology defined by a family of inner products (possibly with non-trivial kernels) is called *Hilbertian.* Such a space is necessarily a projective limit of L^2-spaces [cf. 12, p. 115–116]. Also, every nuclear space is Hilbertian [12, p. 519]. Hence,

THEOREM 5. *Every Hilbertian space, in particular every nuclear space, is L^0-projective.*

For each $1 \leqslant p < \infty$ let l^p be the Banach space of sequences (x_n), $\sum_n | x |^p < \infty$, with the usual norm. Let c_0 be the space of all sequences (x_n), $x_n \to 0$, with the sup norm. We shall now prove:

THEOREM 6. *c_0, and the spaces l^p for $\infty > p > 2$, are not L^0-projective. Any topological vector space with a subspace isomorphic to c_0 or an l^p-space, $\infty > p > 2$, is not L^0-projective.*

Proof. Let $e_n : n = 0, 1,...$ be the canonical basis of c_0, i.e. $e_{nm} = 1, n = m$; $e_{nm} = 0, n \neq m$. From Orlicz [8, Thm. 8(b), p. 300], or Kwapien [6], for any continuous linear process $A: c_0 \to L^0(S, M, v)$ we have that $\sum_n A(e_n)(s)^2 < \infty$ for v-almost all $s \in S$. Thus $e_n \to 0$ in $\mathscr{T}(\lambda^0(c_0))$, but $e_n \not\to 0$ in the norm topology. Hence c_0 is not L^0-projective.

Now, for any $(y_n) \in l^p$,

$$(x_n) \in c_0 \to (x_n y_n) \in l^p$$

is continuous. Hence, as a consequence of the Orlicz–Kwapien theorem, for any $A \in \lambda^0(l^p)$, $\sum_n A(e_n)^2 y_n{}^2$ converges v-almost everywhere. Hence, if $p > 2$, since we must then have $(y_n{}^2) \in l^{p/2}$, it follows that for v-almost $s \in S$, $(A(e_n)(s)^2) \in l^q$, where q is the index conjugate to $p/2$, i.e. $(p/2)^{-1} + q^{-1} = 1$. Since $q < \infty$, then again $e_n \to 0$ in $\mathscr{T}(\lambda^0(l^p))$ but not in the norm topology.

The above theorem applies to suitable infinite-dimensional L^p-spaces, $\infty > p > 2$, and spaces $C(Y)$ of continuous functions.

With little extra effort the theorems given above all extend to the case of complex L^0-spaces and complex vector spaces $X(7)$.

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